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DISTRIBUTION OF PACKET DELAY AND INTERDEPARTURE TIME IN CARRIER--ETC(U)

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(6) DISTRIBUTION OF PACKET DELAY AND  
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IN CARRIER SENSE MULTIPLE ACCESS.

by

(10) Fouad A. Tobagi

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Computer Systems Laboratory  
Stanford Electronics Laboratories  
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Stanford, CA 94305

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Distributions of Packet Delay and Interdeparture Time  
in Carrier Sense Multiple Access\*

Fouad A. Tobagi

Computer Systems Laboratory  
Stanford University  
Stanford, California 94305

Abstract

Existing analysis of the carrier sense multiple access mode (CSMA) has led to the determination of the average channel performance in terms of average throughput and average packet delay. This was achieved by formulating a semi-markovian model for CSMA channels with a finite population of users [1,2]. In this paper, it is shown that, using the same model, it is possible to derive the actual packet delay distribution. The analysis is similar in nature to that provided for slotted ALOHA channels in [3]. These results are useful in the design of systems intended for real time applications such as digitized speech, and in the analysis of multihop packet radio systems.

KEYWORDS: Computer communication networks, Packet switching, Multiple access protocols, Slotted ALOHA, Carrier sense multiple access, Performance, Throughput, Delay.

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## I. INTRODUCTION

Carrier sense multiple access (CSMA) is an efficient random access method for multiplexing a population of users communicating over a shared packet-switched channel. Briefly, CSMA reduces the level of interference (caused by overlapping packets) in the random multi-access environment by allowing terminals to sense the carrier due to other users' transmissions; based on this channel state information (busy or idle), the terminal takes an action prescribed by the particular CSMA protocol in use. In particular, terminals never transmit when they sense that the channel is busy [4].

To analyze the performance of CSMA, a semi-markovian model has been formulated for channels with finite populations of interactive users [1,2]. However only average stationary performance was derived in terms of average throughput and average packet delay. Unfortunately, average performance may not be adequate when designing systems intended for real time applications such as digitized speech, and the analysis has to be extended so as to include delay distributions. Also, when analyzing multihop systems, it is important to be able to characterize the departure process from a collection of nodes, as this corresponds to the arrival process to other nodes in the multihop environment.

In this paper, we extend this analysis and show that, using the same model, one can derive the actual distribution of packet delay and interdeparture time. In section II, we describe

the model. In section III, we review the derivation of the average channel performance as presented in [1,2]. In section IV we address the interdeparture time distribution and its moments. In section V, we treat packet delay.



## II. THE MODEL

Although the operation of CSMA does not require all devices to be time-synchronized, it is assumed here, for simplicity in analysis, that the channel time axis is slotted with the slot size equal to  $\tau$  s., the maximum propagation delay between all pairs of users, and that all users are synchronized to begin transmission only at slot boundaries. The CSMA scheme under consideration here consists of the following. A user with a packet ready for transmission (i.e., with a packet which has just been generated, or which has been rescheduled for transmission at that instant), senses the channel and (i) if the channel is idle, starts transmitting the packet at the beginning of the next slot, and (ii) if the channel is busy, then it reschedules the transmission of the packet to some random time in the future.

We consider a finite population of  $M$  users, all in line of sight and within range of each other, such that each user can be in one or two states: backlogged or thinking. In the thinking state, a user generates a new packet (and starts transmitting the packet, if the channel is sensed idle) in a slot with probability  $\sigma$ . A user is said to be backlogged if it has a packet in transmission or awaiting transmission. It remains in that state until it completes successful transmission of the packet, at which time it switches to the thinking state. Thus a user in the backlogged state cannot generate a new packet for transmission. The rescheduling

delay of a backlogged packet is assumed to be geometrically distributed, i.e., each backlogged user is scheduled to resense the channels in the current slot with probability  $v$ ; as specified by the protocol, a retransmission would result only if the channel is sensed idle. In this model, it is assumed that a user learns about its success or failure instantaneously at the end of its transmission period. All packets are of a fixed size equal to  $T$  slots.

### III. AVERAGE STATIONARY PERFORMANCE [1,2]

Let  $n(t)$  denote the number of backlogged users at the beginning of slot  $t$ . We observe on the time axis an alternate sequence of idle and busy periods as shown in Fig. 1. We follow the approach used in [1,2] and consider the imbedded Markov chain identified by the first slot of each idle period. Using properties of regenerative processes, we derive the average channel performance.

Let  $t_e^{(r)}$  and  $t_e^{(r+1)}$  be two consecutive imbedded slots; the period of time between  $t_e^{(r)}$  and  $t_e^{(r+1)}$  is called a cycle. Let  $\underline{P}$  denote the transition probability matrix between  $t_e^{(r)}$  and  $t_e^{(r+1)}$ ; that is, the  $(i,j)^{th}$  element of  $\underline{P}$  is defined as

$$p_{ij} \triangleq \Pr\{n(t_e^{(r+1)}) = j \mid n(t_e^{(r)}) = i\}. \quad 0 \leq i, j \leq M \quad (1)$$

We let TP denote the length of the transmission period. If the transmission of the message is successful, then  $TP = T + 1$ , where the additional slot accounts for the propagation delay since it is only one slot after the end of transmission that the channel will be sensed idle by all users. If the transmission of the message is unsuccessful, then  $TP = T_c + 1$ , where  $T_c \leq T$  is the time to detect the collision and abort all transmissions if the collision detection feature is in effect, and  $T_c = T$  otherwise.  $n(t_e^{(r)})$  remains invariant over the entire idle period  $I$  (since according to the CSMA procedure, a new arrival sensing the channel

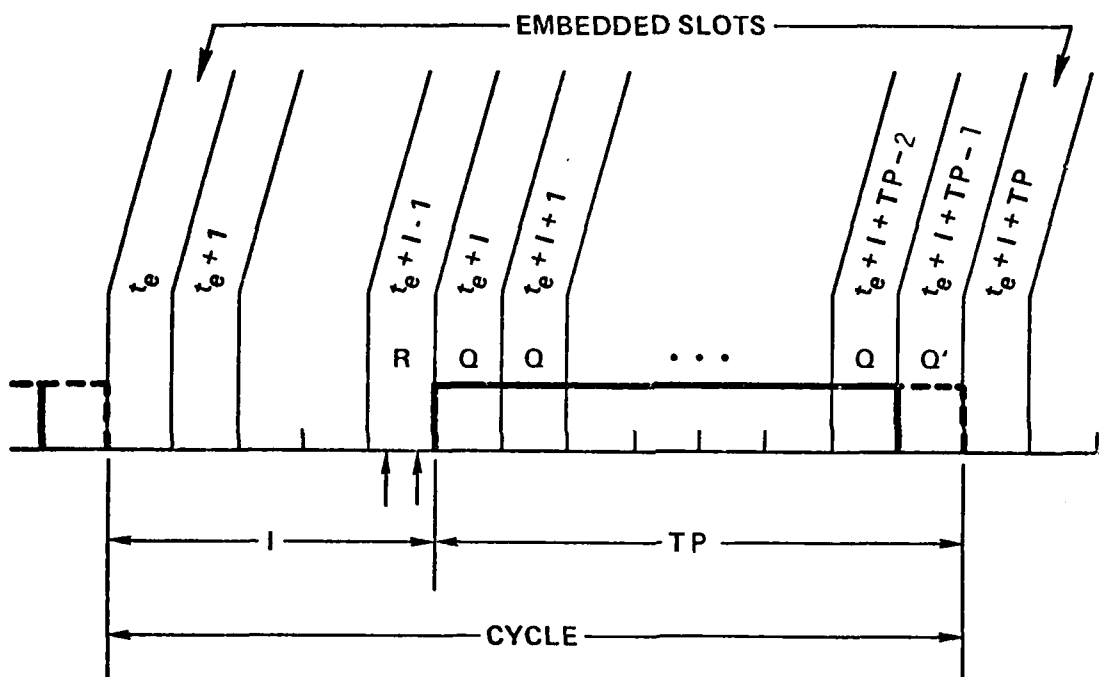


Figure 1. The imbedded Markov chain in CSMA.

idle would transmit with probability one). See Fig. 1. Thus for  $t \in [t_e^{(r)}, t_e^{(r)} + I - 1]$ ,  $n(t) = n(t_e^{(r)})$ . Let  $\underline{R}$  denote the transition matrix between slot  $t_e^{(r)} + I - 1$  and  $t_e^{(r)} + I$ . Since the success or failure of the transmission is a function of the number of users becoming ready in slot  $t_e^{(r)}$ , we write  $\underline{R}$  as  $\underline{R} = \underline{S} + \underline{F}$ , where the  $(i, k)^{\text{th}}$  elements of  $\underline{S}$  and  $\underline{F}$  are defined and expressed as

$$s_{ik} \triangleq \Pr\{n(t_e^{(r)} + I) = k \text{ and transmission is successful} \mid n(t_e^{(r)} + I - 1) = i\}$$

$$= \begin{cases} 0 & k < i \\ \frac{(1-\sigma)^{M-i} [i v (1-v)^{i-1}]}{1 - (1-v)^i (1-\sigma)^{M-i}} & k = i \\ \frac{(M-i) \sigma (1-\sigma)^{M-i-1} (1-v)^i}{1 - (1-v)^i (1-\sigma)^{M-i}} & k = i+1 \\ 0 & k > i+1 \end{cases} \quad (2)$$

$$f_{ik} \triangleq \Pr\{n(t_e^{(r)} + 1) = k \text{ and transmission is unsuccessful}$$

$$| n(t_e^{(r)} + 1) = i\}$$

$$= \begin{cases} 0 & k < i \\ \frac{(1-\sigma)^{M-i} [1-(1-\nu)^i - i\nu(1-\nu)^{i-1}]}{1-(1-\nu)^i (1-\sigma)^{M-i}} & k = i \\ \frac{(M-i)\sigma(1-\sigma)^{M-i-1} [1-(1-\nu)^i]}{1-(1-\nu)^i (1-\sigma)^{M-i}} & k = i+1 \\ \frac{\binom{M-i}{k-i} (1-\sigma)^{M-k} \sigma^{k-i}}{1-(1-\nu)^i (1-\sigma)^{M-i}} & k > i+1 \end{cases}$$

(3)

During the transmission period, all new arrivals join the backlog.

Thus, for any  $t \in [t_e^{(r)} + 1, t_e^{(r)} + 1 + TP]$ , we let  $\underline{Q}$  denote the one-step transition matrix, for which the  $(i, k)^{th}$  element is defined as  $q_{ik} \triangleq \Pr\{n(t) = k \mid n(t-1) = i\}$  and expressed as

$$q_{ik} = \begin{cases} 0 & k < i \\ \binom{M-i}{k-i} (1-\sigma)^{M-k} \sigma^{k-i} & k \geq i \end{cases}$$

(4)

Finally, to represent the fact that a successful transmission decreases the backlog by 1, we introduce matrix  $\underline{J}$  such that

its  $(i,k)^{th}$  element is given by

$$j_{ik} = \begin{cases} 1 & k=i-1 \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

The transition matrix  $\underline{P}$  is then expressed as

$$\underline{P} = \underline{S} \underline{Q}^{T+1} \underline{J} + \underline{F} \underline{Q}^{T_c+1} \quad (6)$$

Let  $\underline{\Pi} = \{\pi_0, \pi_1, \dots, \pi_M\}$  denote the stationary probability distribution of  $n(t_e^{(r)})$ .  $\underline{\Pi}$  is obtained by the recursive solution of  $\underline{\Pi} = \underline{\Pi} \underline{P}$ .

Since  $n(t_e^{(r)})$  is a regenerative process, the average stationary channel throughput is computed as the ratio of the time the channel is carrying successful transmission during a cycle averaged over all cycles, to the average cycle length. Therefore we have

$$S = \frac{\sum_{i=0}^M \pi_i P_s(i) T}{\sum_{i=0}^M \pi_i \left[ \frac{1}{1-\delta_i} + 1 + P_s(i)T + (1-P_s(i))T_c \right]} \quad (7)$$

where  $P_s(i)$  is the probability of a successful transmission during a cycle with  $n(t_e^{(r)}) = i$ , and is given by

$$P_s(i) = \frac{(M-i)\sigma(1-\sigma)^{M-i-1} (1-v)^i + i v (1-v)^{i-1} (1-\sigma)^{M-i}}{1-(1-v)^i (1-\sigma)^{M-i}} \quad (8)$$

and where  $(1-\delta_i)^{-1}$ , with  $\delta_i = (1-v)^i (1-\sigma)^{M-i}$ , is the average idle period given  $n(t_e^{(r)}) = i$ .

Similarly, the average channel backlog is computed as the ratio of the expected sum of backlogs over all slots in a cycle (averaged over all cycles), to the average cycle length. Therefore we have

$$\bar{n} = \frac{\sum_{i=0}^M \pi_i \left( \frac{1}{1-\delta_i} + A(i) \right)}{\sum_{i=0}^M \pi_i \left[ \frac{1}{1-\delta_i} + 1 + P_s(i)T + (1-P_s(i))T_c \right]} \quad (9)$$

where  $A(i)$  is the expected sum of backlogs over all slots in the busy period with  $n(t_e^{(r)}) = i$ , and is given by\*

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\* For an arbitrary matrix  $\underline{B}$ , we adopt the notation  $[\underline{B}]_{ij}$  to represent the  $(i,j)^{th}$  element of  $\underline{B}$ .



$$\begin{aligned}
A(i) &= \sum_{\ell=0}^T \sum_{j=1}^M j [\underline{S} \underline{Q}^{\ell}]_{ij} + \sum_{\ell=0}^{T_c} \sum_{j=1}^M j [\underline{F} \underline{Q}^{\ell}]_{ij} \\
&= \sum_{j=1}^M j \left[ \underline{S} \sum_{\ell=0}^T \underline{Q}^{\ell} + \underline{F} \sum_{\ell=0}^{T_c} \underline{Q}^{\ell} \right]_{ij}
\end{aligned} \tag{10}$$

By Little's result [5], the average packet delay (normalized to T) is simply expressed as

$$\bar{D} = \bar{n}/S \tag{11}$$

#### IV. INTERDEPARTURE TIME DISTRIBUTION

##### Theorem:

The generating function for the interdeparture time distribution is given by

$$ID^*(z) = \underline{A} [\underline{I} - \underline{P}_d^*(z)]^{-1} \underline{P}_d^*(z) \underline{H} \quad (12)$$

where

$$i) \quad \underline{A} \text{ is solution of } \underline{A} = \underline{A} [\underline{I} - \underline{F} \underline{Q}^{T_c+1}]^{-1} \underline{S} \underline{Q}^{T+1} \underline{J}$$

ii)  $\underline{P}_d^*(z)$  and  $\underline{P}_d^*(z)$  are matrices such that their  $(i,j)^{th}$  elements are defined as

$$\left[ \underline{P}_d^*(z) \right]_{ij} = \left[ \underline{S} \underline{Q}^{T+1} \underline{J} \right]_{ij} \frac{(1-\delta_i)z^{T+2}}{1-\delta_i z} \quad (13)$$

$$\left[ \underline{P}_d^*(z) \right]_{ij} = \left[ \underline{F} \underline{Q}^{T_c+1} \right]_{ij} \frac{(1-\delta_i)z^{T_c+2}}{1-\delta_i z} \quad (14)$$

$$\delta_i = (1-\nu)^i (1-\sigma)^{M-1} \quad (15)$$

iii)  $\underline{H}$  is a column vector with all elements equal to one.

Proof:

The outline of the proof is similar to that given in [3] for slotted ALOHA channels.

Consider an imbedded slot  $t_e^{(r)}$  such that  $n(t_e^{(r)}) = 1$ .

Let  $ID_1^*(z)$  denote the generating function of the distribution of time until completion of the first successful transmission following  $t_e^{(r)}$ . Let  $I_1^*(z)$  denote the generating function of the distribution of the idle period. Since the latter is geometrically distributed with mean  $1/(1-\delta_1)$ , where  $\delta_1 = (1-\nu)^1 (1-\sigma)^{M-1}$ ,  $I_1^*(z)$  is given by

$$I_1^*(z) = \frac{(1-\delta_1)z}{1-\delta_1 z} \quad (16)$$

$ID_1^*(z)$  is just  $I_1^*(z)z^{T+1}$  if the first transmission is successful, and  $I_1^*(z)z^{T_c+1} ID_j^*(z)$  if the first transmission is unsuccessful and  $n(t_e^{(r+1)}) = j$ . Thus letting  $p_{ij}^{(d)} \triangleq [S Q^{T+1} J]_{ij} \triangleq [P_d]_{ij}$  and  $p_{ij}^{(\bar{d})} \triangleq [F Q^{T_c+1}]_{ij} \triangleq [P_{\bar{d}}]_{ij}$ , we have

$$ID_1^*(z) = \sum_{j=1}^M \left[ p_{ij}^{(d)} I_1^*(z)z^{T+1} + p_{ij}^{(\bar{d})} I_1^*(z)z^{T_c+1} ID_j^*(z) \right] \quad (17)$$

Let  $\underline{ID}^*(z)$  denote the column vector  $(ID_0^*(z), \dots, ID_M^*(z))^T$  (the superscript T representing the transpose operation); let  $\underline{P}_d^*(z)$  and  $\underline{P}_d^*(z)$  be as defined in Eqs. (13,14); we can rewrite Eq. (16) in matrix notation as

$$\underline{ID}^*(z) = \underline{P}_d^*(z)\underline{H} + \underline{P}_d^*(z)\underline{ID}^*(z) \quad (18)$$

or

$$\underline{ID}^*(z) = [\underline{I} - \underline{P}_d^*(z)]^{-1} \underline{P}_d^*(z)\underline{H} \quad (19)$$

To obtain  $ID^*(z)$ , we need to remove the condition on  $n(t_e^{(r)})$ . Let  $\dots t_d^{(r)}, t_d^{(r+1)}, \dots$  denote the sequence of imbedded points immediately following a successful transmission.  $\{n(t), t \in (\dots, t_d^{(r)}, t_d^{(r+1)}, \dots)\}$  is an imbedded Markov chain with transition probabilities given by

$$\begin{aligned} \Pr\{n(t_d^{(r+1)}) = j \mid n(t_d^{(r)}) = i\} &= \sum_{\ell=1}^{\infty} \left[ \underline{P}_d^{\ell-1} \underline{P}_d \right]_{ij} \\ &= \left[ (\underline{I} - \underline{P}_d)^{-1} \underline{P}_d \right]_{ij} \end{aligned} \quad (20)$$

The stationary distribution of  $n(t_d^{(r)})$ ,  $\underline{\Delta} = (d_0, d_1, \dots, d_M)$ , is solution of  $\underline{\Delta} = \underline{\Delta} (\underline{I} - \underline{P}_d)^{-1} \underline{P}_d$ . We finally have  $ID^*(z) = \underline{\Delta} \underline{ID}^*(z)$ , hence Eq. (12).

Q.E.D.

A simple recursive procedure exists for the computation of the  $m^{\text{th}}$  moment of ID. Let  $ID^{(m)} \triangleq \left. \frac{d^m ID^*(z)}{dz^m} \right|_{z=1}$ ,  $ID^*(m)(z) \triangleq \frac{d^m ID^*(z)}{dz^m}$ ,  $\underline{P}_d^{*(m)}(z) \triangleq \frac{d^m \underline{P}_d^*(z)}{dz^m}$  and  $\underline{P}_d^{*(m)}(z) \triangleq \frac{d^m \underline{P}_d^*(z)}{dz^m}$ . (We use the convention

that the derivative of a vector or matrix is the vector or matrix whose elements are derivatives of the corresponding elements in the original vector or matrix). Clearly  $E[ID] = ID^{(1)}$ ;  $\text{Var}[ID] = ID^{(2)} + ID^{(1)} - [ID^{(1)}]^2$ ,  $ID^{(3)} = E[ID(ID-1)(ID-2)]$ , etc.

Corollary:

$ID^{(m)}$  is given by

$$ID^{(m)} = \underline{A} \cdot ID^{*(m)}(1) \quad (21)$$

where  $ID^{*(m)}(1)$  is recursively determined by

$$ID^{*(m)}(1) = (\underline{I} - \underline{P}_d)^{-1} \left[ \underline{P}_d^{*(m)}(1) \cdot \underline{H} + \underline{P}_d^{*(m)}(1) \cdot \underline{H} + \sum_{k=1}^{m-1} \binom{m}{k} \underline{P}_d^{*(k)}(1) \cdot ID^{*(m-k)}(1) \right] \quad (22)$$

Proof:

It can be easily proven by induction that differentiating Eq. (18)  $m$  times leads to the following relation

$$\underline{ID}^{*(m)}(z) = \underline{P}_d^{*(m)}(z) \underline{H} + \sum_{k=0}^m \binom{m}{k} \underline{P}_d^{*(k)}(z) \cdot \underline{ID}^{*(m-k)}(z) \quad (23)$$

Letting  $z=1$  and observing that  $\underline{ID}^*(1) = \underline{H}$ , and  $\underline{P}_d^*(1) = \underline{P}_d$ , we get Eq. (22).

Q.E.D.

The average interdeparture time, in particular, is given by

$$ID^{(1)} = \underline{A} (\underline{I} - \underline{P}_d)^{-1} \left[ \underline{P}_d^{*(1)}(1) + \underline{P}_d^{*(1)}(1) \right] \underline{H} \quad (24)$$

where  $\left[ \underline{P}_d^{*(1)}(1) + \underline{P}_d^{*(1)}(1) \right] \underline{H}$  is a column vector whose  $i^{th}$  element is simply

$$\frac{1}{1-\delta_i} + \sum_{j=i-1}^M \left[ p_{ij}^{(d)}(T+1) + p_{ij}^{(\bar{d})}(T_c+1) \right]$$

The variance of ID is given by

$$\begin{aligned} \text{Var}[ID] = \underline{A} (\underline{I} - \underline{P}_d)^{-1} & \left[ \left[ \underline{P}_d^{*(2)}(1) + \underline{P}_d^{*(2)}(1) \right] \underline{H} + 2 \underline{P}_d^{*(1)}(1) \underline{ID}^{*(1)}(1) \right] \\ & + ID^{(1)} - [ID^{(1)}]^2 \end{aligned} \quad (25)$$

## V. DISTRIBUTION OF PACKET DELAY

Consider an imbedded slot  $t_e^{(r)}$  such that  $n(t_e^{(r)}) = i$ , and let  $D_i^*(z)$  denote the  $z$ -transform of the distribution of delay (counted starting from  $t_e^{(r)}$ ) until a tagged user in the backlog of size  $i$  is successful. Let  $\underline{D}^*(z)$  be the column vector  $(D_1^*(z), \dots, D_M^*(z))^T$ .

### Theorem:

$\underline{D}^*(z)$  is given by

$$\underline{D}^*(z) = [\underline{I} - \underline{P}_S^*(z)]^{-1} \underline{P}_S^*(z) \underline{H} \quad (26)$$

where  $\underline{H}$  is the column vector with all elements equal to one, and

$\underline{P}_S^*(z)$  and  $\underline{P}_F^*(z)$  are matrices with  $(i,j)^{th}$  elements defined as

$$[\underline{P}_S^*(z)]_{ij} = \left[ \underline{S} \underline{Q}^{T+1} \underline{J} \right]_{ij} \frac{(1-\delta_i)z^{T+2}}{1-\delta_i z} \quad (27)$$

$$[\underline{P}_F^*(z)]_{ij} = \left[ \underline{S} \underline{Q}^{T+1} \underline{J} \right]_{ij} \frac{(1-\delta_i)z^{T+2}}{1-\delta_i z} + \left[ \underline{F} \underline{Q}^{T_c+1} \right]_{ij} \frac{(1-\delta_i)z^{T_c+2}}{1-\delta_i z} \quad (28)$$

$$[\underline{S}]_{ik} = \begin{cases} \frac{1}{i} [\underline{S}]_{i,i} & k=i \\ 0 & \text{otherwise} \end{cases} \quad (29)$$

$$\underline{S}_s = \underline{S} - \underline{S}_s \quad (30)$$

Proof:

The proof is similar to that given in Section III for  $\underline{ID}^*(z)$ . Noting that

$$[\underline{S}_s]_{ik} = \Pr\{n(t_e^{(r)} + I + 1) = k \text{ and tagged user successful} \mid n(t_e^{(r)}) = i\}$$

we have

$$\begin{aligned} D_i^*(z) &= \sum_{j=i-1}^M [\underline{S}_s \underline{Q}^{T+1} \underline{J}]_{ij} \frac{(1-\delta_i)z^{T+2}}{1-\delta_i z} \\ &+ \sum_{j=i-1}^M \left[ [\underline{S}_s \underline{Q}^{T+1} \underline{J}]_{ij} \frac{(1-\delta_i)z^{T+2}}{1-\delta_i z} + [\underline{F} \underline{Q}^{T_c+1}]_{ij} \frac{(1-\delta_i)z^{T_c+2}}{1-\delta_i z} \right] D_j^*(z) \end{aligned} \quad (31)$$

hence Eq. (26).

Q.E.D.

Theorem:

The distribution of delay is given by

$$D^*(z) = \xi_0^{(T+1)} z^{T+1} + \sum_{\ell=0}^T z^\ell \underline{\xi}^{(\ell)} \underline{D}^*(z) \quad (32)$$

where  $\xi_0^{(T+1)}$  and  $\underline{\xi}^{(\ell)}$  are defined in Eqs. (49-53) in the proof.



Proof:

To complete the delay calculation, we need to compute  $\xi_k^{(\ell)}$ , the probability that an arbitrary new packet arrives in a slot which is  $\ell$  slots away from the next imbedded slot, and finds itself at the beginning of this imbedded slot in a backlog of size  $k$ ; indeed, its delay is then  $z^\ell D_k^*(z)$ .

Let  $t_e$  and  $t'_e$  denote two successive imbedded slots  $t'_e = t_e + I + TP + 1$ . Let  $\alpha(t_e)$  be an indicator such that

$$\alpha(t_e) = \begin{cases} 0 & \text{if no new arrivals to the backlog occurred} \\ & \text{in the cycle } [t_e, t'_e-1] \\ 1 & \text{if at least one new arrival to the backlog} \\ & \text{occurred in the cycle } [t_e, t'_e-1] \end{cases} \quad (33)$$

We denote by  $t_a$  an imbedded slot such that  $\alpha(t_a) = 1$ . We first seek the stationary distribution of  $n(t_a)$ , that is the distribution of backlog at the beginning of a cycle in which arrivals occurred.

Let  $t_a^{(r)}$  and  $t_a^{(r+1)}$  denote two successive imbedded slots

such that  $\alpha(t_a^{(r)}) = \alpha(t_a^{(r+1)}) = 1$ . The imbedded process

$\{n(t_a), t_a = \dots, t_a^{(r)}, t_a^{(r+1)}, \dots\}$  is a Markov chain with transition

probabilities  $\Pr(n(t_a^{(r+1)}) = j \mid n(t_a^{(r)}) = i)$  derived as follows.

Let  $\underline{S}_a$ ,  $\underline{F}_a$ ,  $\underline{Q}_a$ ,  $\underline{P}_a$  and  $\underline{P}_a$  be transition probability matrices defined as

$$s_{ik}^{(\bar{a})} \triangleq [S_{\bar{a}}]_{ik} \triangleq \Pr(n(t_e + I + 1) = k; \text{ transmission is successful; } \\ \text{no new arrivals in } t_e + I \mid n(t_e + I) = i\}$$

$$= \begin{cases} s_{i,i} & k=i \\ 0 & \text{otherwise} \end{cases} \quad (34)$$

$$f_{ik}^{(\bar{a})} \triangleq [F_{\bar{a}}]_{ik} \triangleq \Pr(n(t_e + I + 1) = k; \text{ transmission is unsuccessful; } \\ \text{no new arrivals in } t_e + I \mid n(t_e + I) = i\}$$

$$= \begin{cases} f_{i,i} & k=i \\ 0 & \text{otherwise} \end{cases} \quad (35)$$

$$q_{ik}^{(\bar{a})} \triangleq [Q_{\bar{a}}]_{ik} = \begin{cases} q_{i,i} = (1-\sigma)^{M-i} & k=i \\ 0 & \text{otherwise} \end{cases} \quad (36)$$

$$\underline{P}_{\bar{a}} \triangleq \underline{S}_{\bar{a}} \underline{Q}_{\bar{a}}^{T+1} \underline{J} + \underline{F}_{\bar{a}} \underline{Q}_{\bar{a}}^{T_c+1} \quad (37)$$

$$\underline{P}_{\bar{a}} \triangleq \underline{P} - \underline{P}_{\bar{a}} \quad (38)$$

It is clear from the above that

$$[\underline{P}_{\bar{a}}]_{ij} = \Pr(n(t'_e) = j, \alpha(t_e) = 0 \mid n(t_e) = i) \quad (39)$$

$$[\underline{P}_a]_{ij} = \Pr\{n(t'_e) = j, \alpha(t_e) = 1 \mid n(t_e) = i\} \quad (40)$$

$$[\underline{P}_a]_{ij} = \begin{cases} \frac{(1-\sigma)^{(M-1)(T+2)} [iv(1-v)^{i-1}]}{1-(1-v)^i (1-\sigma)^{M-i}} & j=i-1 \\ \frac{(1-\sigma)^{(M-1)(T_c+2)} [1-(1-v)^i - iv(1-v)^{i-1}]}{1-(1-v)^i (1-\sigma)^{M-i}} & j=i \\ 0 & \text{otherwise} \end{cases} \quad (41)$$

$$\gamma_i \triangleq \Pr\{\alpha(t_e) = 0 \mid n(t_e) = i\} = [\underline{P}_a]_{i,i-1} + [\underline{P}_a]_{i,i} \quad (42)$$

$$\Pr\{\alpha(t_e) = 1 \mid n(t_e) = i\} = 1 - \gamma_i \quad (43)$$

Let  $\underline{P}_{11}$ ,  $\underline{P}_{10}$ ,  $\underline{P}_{00}$ , and  $\underline{P}_{01}$  be transition probability matrices defined as:

For  $i = 0, 1, \dots, M-1$ ;  $j = 0, 1, \dots, M$ ,

$$\begin{aligned} [\underline{P}_{11}]_{ij} &\triangleq \Pr\{n(t'_e) = j, \alpha(t'_e) = 1 \mid n(t_e) = i, \alpha(t_e) = 1\} \\ &= [\underline{P}_a]_{ij} \frac{1-\gamma_j}{1-\gamma_i} \end{aligned} \quad (44)$$

$$\begin{aligned}
[\underline{P}_{10}]_{ij} &\stackrel{\Delta}{=} \Pr\{n(t'_e) = j, \alpha(t'_e) = 0 \mid n(t_e) = i, \alpha(t_e) = 1\} \\
&= [\underline{P}_a]_{ij} \frac{\gamma_j}{1-\gamma_i}
\end{aligned} \tag{45}$$

For  $i = 0, 1, \dots, M; j = 0, 1, \dots, M$

$$\begin{aligned}
[\underline{P}_{00}]_{ij} &\stackrel{\Delta}{=} \Pr\{n(t'_e) = j, \alpha(t'_e) = 0 \mid n(t_e) = i, \alpha(t_e) = 0\} \\
&= [\underline{P}_a]_{ij} \frac{\gamma_j}{\gamma_i}
\end{aligned} \tag{46}$$

$$\begin{aligned}
[\underline{P}_{01}]_{ij} &\stackrel{\Delta}{=} \Pr\{n(t'_e) = j, \alpha(t'_e) = 1 \mid n(t_e) = i, \alpha(t_e) = 0\} \\
&= [\underline{P}_a]_{ij} \frac{1-\gamma_j}{\gamma_i}
\end{aligned} \tag{47}$$

It is clear that for  $i, j \neq M$  we have

$$\begin{aligned}
\Pr\{n(t_a^{(r+1)}) = j \mid n(t_a^{(r)}) = i\} \\
&= [\underline{P}_{11}]_{ij} + [\underline{P}_{10}(I + \underline{P}_{00} + \underline{P}_{00}^2 + \dots) \underline{P}_{01}]_{ij} \\
&= [\underline{P}_{11}]_{ij} + [\underline{P}_{10} (I - \underline{P}_{00})^{-1} \underline{P}_{01}]_{ij}
\end{aligned} \tag{48}$$

Let  $\underline{A}$  denote the above transition matrix restricted to rows

$i = 0, 1, \dots, M-1$  and columns  $j = 0, 1, \dots, M-1$ . Let  $\underline{\alpha} = \{\alpha_0, \dots, \alpha_{M-1}\}$  denote the stationary distribution of  $n(t_a^{(r)})$ .  $\underline{\alpha}$  is obtained as the solution of  $\underline{\alpha} = \underline{\alpha} \underline{A}$ .

Consider an imbedded slot  $t_a^{(r)}$  such that  $n(t_a^{(r)}) = i < M$  and  $\alpha(t_a^{(r)}) = 1$ . Let  $t_e'$  denote the imbedded slot following  $t_a^{(r)}$ . Let  $\xi_k^{(\ell)}(i)$  denote the probability that a tagged packet generated in the cycle  $[t_a^{(r)}, t_e' - 1]$  arrives in slot  $t_e' - \ell - 1$  and finds itself at the beginning of  $t_e'$  in a backlog of size  $k$ . We use the value " $k = 0$ " to represent the case where the arriving packet starts successful transmission at its arrival; clearly in this case the new packet must have arrived in slot  $t_a^{(r)} + 1$ , and  $\ell$  must equal  $T + 1$ . We now proceed with the derivation of  $\xi_k^{(\ell)}(i)$ .

Let  $m_\ell$ ,  $0 \leq m_\ell \leq M-i$ , denote the number of arrivals in slot  $t_e' - \ell - 1$ . If the transmission is successful, then  $0 \leq \ell \leq T+1$ ; we define in this case  $\mu_v$  to be

$$\mu_v \triangleq \sum_{j=T+1-v}^{T+1} m_j \quad 0 \leq v \leq T+1$$

If the transmission is unsuccessful, then  $0 \leq \ell \leq T_c+1$ ; we define in this case  $\mu'_v$  to be

$$\mu'_v \triangleq \sum_{j=T_c+1-v}^{T_c+1} m_j \quad 0 \leq v \leq T_c+1$$

For  $\mu_{T+1} \leq M-i$  and  $\mu'_{T_c+1} \leq M-i$ , we have

$\Pr\{(m_0, m_1, \dots, m_{T+1}), \text{ transmission successful}$

$$| n(t_a^{(r)}) = i, \alpha(t_a^{(r)}) = 1\}$$

$$= \frac{[S]_{i, i+\mu_0} \prod_{v=0}^T [Q]_{i+\mu_v, i+\mu_{v+1}}}{1-\gamma_i} \quad (49)$$

$\Pr\{(m_0, m_1, \dots, m_{T_c+1}), \text{ transmission unsuccessful}$

$$| n(t_a^{(r)}) = i, \alpha(t_a^{(r)}) = 1\}$$

$$= \frac{[F]_{i, i+\mu'_0} \prod_{v=0}^{T_c} [Q]_{i+\mu'_v, i+\mu'_{v+1}}}{1-\gamma_i} \quad (49a)$$

It is also clear that for  $0 \leq \ell \leq T+1$  and  $0 \leq k \leq M$ , we have

$\Pr\{\text{tagged user arrives in slot } t'_e - \ell - 1 \text{ and finds itself}$   
in a backlog of size  $k \mid (m_0, m_1, \dots, m_{T+1}),$   
transmission successful,  $n(t_a^{(r)})=i, \alpha(t_a^{(r)})=1\}$

$$= \begin{cases} \frac{1}{\mu_{T+1}} & \text{if } \ell=T+1, (k=0), m_{T+1}=1, 1 \leq \mu_{T+1} \leq M-1 \\ \frac{m_\ell}{k-i+1} & \text{if } 0 \leq \ell \leq T, i \leq k \leq M, \mu_{T+1}=k-i+1 \\ 0 & \text{otherwise} \end{cases} \quad (50)$$

Similarly, for  $0 \leq \ell \leq T_c+1$  and  $1 \leq k \leq M$ , we have

$$\begin{aligned} & \Pr\{\text{tagged user arrives in slot } t'_e - \ell - 1 \text{ and finds itself} \\ & \quad \text{in a backlog of size } k \mid (m_0, m_1, \dots, m_{T_c+1}), \\ & \quad \text{transmission unsuccessful, } n(t_a^{(r)})=i, \alpha(t_a^{(r)})=1\} \\ &= \begin{cases} \frac{m_\ell}{k-i} & \text{if } 0 \leq \ell \leq T_c+1, i+1 \leq k \leq M, \mu'_{T_c+1}=k-i \\ 0 & \text{otherwise} \end{cases} \quad (50a) \end{aligned}$$

Therefore we can write

$$\begin{aligned} \xi_0^{(T+1)}(i) &= \sum_{\substack{(m_0, \dots, m_T, 1) \\ \mu_{T+1} \leq M-i}} \frac{1}{\mu_{T+1}} \cdot \frac{[S]_{i,i+1} \prod_{v=0}^T [Q]_{i+\mu_v, i+\mu_{v+1}}}{1-\gamma_i} \\ &= \sum_{\mu_{T+1}=1}^{M-i} \frac{1}{\mu_{T+1}} \cdot \frac{[S]_{i,i+1} [Q^{T+1}]_{i+1, i+\mu_{T+1}}}{1-\gamma_i} \quad (51) \end{aligned}$$

and for  $i \leq k \leq M$

$$\varepsilon_k^{(\ell)}(i) = \left\{ \begin{array}{l} \sum_{\substack{(m_0, \dots, m_{T+1}) \\ \mu_{T+1} = k-i+1}} \frac{m_\ell}{k-i+1} \frac{[S]_{i, i+\mu_0} \prod_{v=0}^T [Q]_{i+\mu_v, i+\mu_{v+1}}}{1-\gamma_i} \\ \quad \text{if } T_c+2 \leq \ell \leq T \text{ and } i \leq k \leq M \\ \quad \text{and if } 0 \leq \ell \leq T_c+1 \text{ and } k=i \\ \sum_{\substack{(m_0, \dots, m_{T+1}) \\ \mu_{T+1} = k-i+1}} \frac{m_\ell}{k-i+1} \frac{[S]_{i, i+\mu_0} \prod_{v=0}^T [Q]_{i+\mu_v, i+\mu_{v+1}}}{1-\gamma_i} \\ + \sum_{\substack{(m_0, \dots, m_{T_c+1}) \\ \mu'_{T_c+1} = k-i}} \frac{m_\ell}{k-i} \frac{[F]_{i, i+\mu'_0} \prod_{v=0}^{T_c} [Q]_{i+\mu'_v, i+\mu'_{v+1}}}{1-\gamma_i} \\ \quad \text{if } 0 \leq \ell \leq T_c+1 \text{ and } i+1 \leq k \leq M \end{array} \right.$$

which reduces to



$$\begin{aligned}
\xi_k^{(\ell)}(i) = & \left\{ \begin{aligned}
& \sum_{m_\ell=1}^{k-i+1} \frac{m_\ell}{k-i+1} \sum_{x=i}^{k+1-m_\ell} \frac{[S \ Q^{T-\ell}]_{i,x} [Q]_{x,x+m_\ell} [Q^\ell J]_{x+m_\ell,k}}{1-\gamma_i} \\
& \quad \text{if } T_c+2 \leq \ell \leq T \text{ and } i \leq k \leq M \\
& \quad \text{and if } 0 \leq \ell \leq T_c+1 \text{ and } k=i \\
& \sum_{m_\ell=1}^{k-i+1} \frac{m_\ell}{k-i+1} \sum_{x=i}^{k+1-m_\ell} \frac{[S \ Q^{T-\ell}]_{i,x} [Q]_{x,x+m_\ell} [Q^\ell J]_{x+m_\ell,k}}{1-\gamma_i} \\
& + \sum_{m_\ell=1}^{k-i} \frac{m_\ell}{k-i} \frac{[F]_{i,i+m_\ell} [Q^{T_c+1}]_{i+m_\ell,k}}{1-\gamma_i} \\
& \quad \text{if } \ell=T_c+1 \text{ and } i+1 \leq k \leq M \\
& \sum_{m_\ell=1}^{k-i+1} \frac{m_\ell}{k-i+1} \sum_{x=i}^{k+1-m_\ell} \frac{[S \ Q^{T-\ell}]_{i,x} [Q]_{x,x+m_\ell} [Q^\ell J]_{x+m_\ell,k}}{1-\gamma_i} \\
& + \sum_{m_\ell=1}^{k-1} \frac{m_\ell}{k-i} \sum_{x=i}^{k-m_\ell} \frac{[F \ Q^{T_c-\ell}]_{i,x} [Q]_{x,x+m_\ell} [Q^\ell]_{x+m_\ell,k}}{1-\gamma_i} \\
& \quad \text{if } 0 \leq \ell \leq T_c \text{ and } i+1 \leq k \leq M
\end{aligned} \right.
\end{aligned}$$

(52)

We remove the condition on  $n(t_a^{(r)})$  by noting that  $a_i = \Pr\{n(t_a^{(r)})=i \mid \alpha(t_a^{(r)})=1\}$ , and thus write

$$\xi_k^{(\ell)} = \sum_{i=0}^{M-1} a_i \xi_k^{(\ell)}(i) \quad (53)$$

Let  $\underline{\xi}^{(\ell)} = (\xi_1^{(\ell)}, \xi_2^{(\ell)}, \dots, \xi_M^{(\ell)})$ . The distribution of packet delay is  $z^{T+1}$  with probability  $\xi_0^{(T+1)}$ , and  $z^{\ell} \underline{D}_k^*(z)$  with probability  $\xi_k^{(\ell)}$ ; hence Eq. (32).

Q.E.D.

As with interdeparture times, a simple recursive procedure exists for the computation of all moments of the delay.

$$\text{Let } D^{(m)} \triangleq \left. \frac{d^m D^*(z)}{dz^m} \right|_{z=1}, \quad \underline{D}^{*(m)}(z) \triangleq \frac{d^m \underline{D}^*(z)}{dz^m}, \quad \underline{P}_s^{*(m)}(z) \triangleq \frac{d^m \underline{P}_s^*(z)}{dz^m}$$

$$\text{and } \underline{P}_s^{*(m)}(z) \triangleq \frac{d^m \underline{P}_s^*(z)}{dz^m}.$$

Corollary:

$D^{(m)}$  is given by

$$D^{(m)} = \frac{(T+1)!}{(T+1-m)!} \xi_0^{(T+1)} + \sum_{\ell=0}^T \left[ \sum_{k=0}^{\min(\ell, m)} \binom{m}{k} \frac{\ell!}{(\ell-k)!} \underline{\xi}^{(\ell)} \underline{D}^{*(m-k)}(1) \right] \quad (54)$$

where  $\underline{D}^{*(m)}(1)$  is recursively determined by

$$\underline{D}^{*(m)}(1) = [\underline{I} - \underline{P}_s^*(1)]^{-1} \left[ \underline{P}_s^*(m)(1)\underline{H} + \underline{P}_s^*(m)(1)\underline{H} + \sum_{k=1}^{m-1} \binom{m}{k} \underline{P}_s^*(k)(1) \underline{D}^{*(m-k)}(1) \right] \quad (55)$$

Proof:

By successive differentiations of Eq. (31) and Eq. (32), and letting  $z=1$ , we can easily establish Eqs. (54,55).

Q.E.D.

## VI. CONCLUSION

We derived in this paper the distributions of packet delay and interdeparture time for CSMA channels with finite populations of interactive users. The approach is similar in nature to that used for slotted ALOHA in [3], except that the basic model underlying the analysis is here semi-markovian rather than markovian. We have derived simple recursive procedures to compute all moments of packet delay and interdeparture time, and closed form expressions for their generating functions.

Contrary to their counterpart in slotted ALOHA, the generating functions derived for CSMA may not prove very practical in the numerical computation of the distributions. Indeed, they require symbolic inversion of matrices whose elements are themselves  $z$ -transforms. However, a numerical procedure which allows to compute approximations to the distributions can be devised as follows. Consider first the case of interdeparture time. It is easily realized that

$$\underline{ID}^*(z) = \sum_{m=1}^{\infty} \underline{ID}_{(m)}^*(z) \quad (56)$$

where

$$\underline{ID}_{(m)}^*(z) \triangleq \left( ID_{0,(m)}^*(z), \dots, ID_{M,(m)}^*(z) \right) \quad (57)$$

$$\underline{ID}_{(m)}^* = \frac{P_d^*}{d}(z) \underline{ID}_{(m-1)}^*(z) \quad \text{for } m \geq 2 \quad (58)$$

$$\underline{ID}_{(1)}^* = \underline{P}_d^*(z) \underline{H} \quad (59)$$

Eqs. (58,59) are equivalent to

$$ID_{i,(1)}^*(z) = \sum_{j=i-1}^M \left[ \underline{P}_d \right]_{ij} (1-\delta_i) z^{T+2} \left[ 1 + \delta_i z + \delta_i^2 z^2 + \dots \right] \quad (60)$$

and for  $m \geq 2$

$$ID_{i,(m)}^*(z) = \sum_{j=i-1}^M \left[ \underline{P}_d \right]_{ij} (1-\delta_i) z^{T_c+2} \left[ 1 + \delta_i z + \delta_i^2 z^2 + \dots \right] ID_{j,(m-1)}^*(z) \quad (61)$$

Thus by successive polynomial multiplications and additions one can generate numerically an approximation of the distribution of ID, the accuracy of which is a function of the position at which the infinite series are truncated and of the maximum value given to  $m$ . A similar procedure can be devised for the distribution of delay.

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